MODELLING EXCESS PROFIT

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Abstract. This paper deals with the problem of modelling in a formal way the concept of excess profit, also known as residual income. A common idea is that excess profit is an unequivocal concept, being the difference between profit and costs, where all types of costs are taken into account, included the opportunity cost, i.e. the profit the entrepreneur would obtain if she invested in another business. This paper aims at showing that this difference is not univocal and that different approaches may be followed to give voice to such a notion. It turns out that two different interpretations are possible. The one existing in the literature is well described by Preinrich (1938), Edwards and Bell (1961) and, more recently, by Peasnell (1981, 1982) in the accounting literature and by Stewart (1991) in the value-based management literature. The interpretation here provided gives rise to a different way of modelling the notion of excess profit. While the existing models are tied to the financial literature, the model here presented is more akin to a microeconomic perspective. The paper focuses on the formal relations among the various models and necessary and sufficient conditions are provided for the integration of all models in the *systemic* framework here adopted. Furthermore, it shows that the systemic paradigm enjoys an aggregation property which makes residual incomes aggregate in a value sense and enables one to reduce forecasting errors in valuation.

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Introduction

The concepts of excess profit and opportunity cost are fundamental notions in economics (and in the conduct of daily life as well). Many synonyms have been coined in the literature to mean 'excess profit': 'excess realizable profit' (Edwards and Bell, 1961), 'excess income' (Kay, 1976), 'abnormal earnings' (Peasnell, 1981), 'supernormal profit' (Begg, Fischer, & Dornbusch, 1984, p.121). 'Economic profit' is also a common term in economic theory, whereas 'residual income' is common in the accounting literature and applied corporate finance. The concept of 'Goodwill' (Preinrich, 1936) is also strictly related to that of excess profit.

How to model residual income from a formal point of view is a problem that has to do with the interpretation of this notion. Preinrich (1938) and Edwards and Bell (1961)) provide a formal representation of excess profit and, in recent years, a renewed interest in the issue is shown in the contributions of Peasnell (1981, 1982) and Stewart (1991) and of some authors whose works address the problem of decomposing a cash-flow stream (Gronchi, 1986, 1987; Peccati, 1987, 1989; Pressacco and Stucchi, 1997). All of the above-mentioned models share the same perspective, that is the same way in which the notion of excess profit is interpreted. This paper aims at showing that a different interpretation is possible, leading to a different model. On the basis of Magni (2003) a formal approach to the notion of excess profit is here provided, and a new index is presented, here named *Systemic Value Added* (henceforth SVA), which bears interesting relations to the classical models.

In the sequel I shall rest (among others) on the following assumptions, unless otherwise specified: An economic agent aims at evaluating the periodic performance of an economic activity P (e.g. a firm, a project), with capital $a_0>0$ at time 0, and it is assumed that equidistant cash flows $a_s \in \mathbb{R}$ are available at time $s=1,2,...,n$. All flows are certain. The business is equity-financed so that the net operating profit coincides with the net profit (such an assumption is uninfluential but simplifies notation).

1. Definition of excess profit and formal translation

The owner of a firm with capital equal to a_0 can be seen as an investor undertaking a business P with initial outlay $-a_0$. Assume the owner of the business can alternatively invest the same capital in a business C (henceforth often called "account C") and that the cash flows released by P are reinvested in (or withdrawn from) account C, then the latter's value evolves according to the following recurrence equation:

$$C_s = C_{s-1}(1+i) + a_s \qquad s = 1, \ldots, n$$

where i is the so-called *opportunity cost of capital*, measuring the rate of return of the alternative business C. I shall assume that, prior to the decision, the value of C is given by

$C_0^*=E_0$, where E_0 denotes the value of the evaluator's wealth at time 0, $E_0 \in \mathbb{R}$. For evaluating business P, the owner of the business compares two alternative lines of action:

(i) to invest in business P,[1]
(ii) to invest the same capital in account C.

The alternative (ii) is evidently the opportunity cost of investing in P.
Let us denote with E_s and E^s, $s \geq 1$, the wealth of the entrepreneur at time s for case (i) and (ii) respectively.

Definition 1.1. *The Net Final Value* (NFV) *of a business P is given by*

$$\text{NFV} = \big(E_n - E_0\big) - \big(E^n - E_0\big) = E_n - E^n.$$

The Net Present Value (NPV) *of a project P is given by*

$$\text{NPV} = \text{NFV}(1+i)^{-n}.$$

In our case we have

$$\begin{aligned} \text{NFV} &= \Big[(E_0 - a_0)\,(1+i)^n + \sum_{s=1}^{n} a_s(1+i)^{n-s}\Big] - \Big[E_0(1+i)^n\Big] \\ &= -a_0(1+i)^n + \sum_{s=1}^{n} a_s(1+i)^{n-s}. \end{aligned} \tag{1}$$

Assuming that P belongs to the class of Soper (1959) we have the following:

Definition 1.2. *The outstanding capital (or business balance) of P at time s is*

$$\begin{aligned} w_0 &= a_0 \\ w_s &= w_{s-1}(1+x) - a_s \qquad s = 1, 2, \ldots, n. \end{aligned} \tag{2a}$$

where x is the internal rate of return.

Assume that P is partly financed with debt (e.g. a loan contract) whose cash flows are $f_0>0$ and $-f_s<0$ for $s \geq 1$ we have the following:

[1] Note that if (i) is chosen, then the value of C at time 0, just after the decision has been taken, becomes

$$C_0 = C_0^* - a_0 = E_0 - a_0$$

as a_0 is withdrawn from account C.

Definition 1.3. *The outstanding debt (or residual debt) at time s is*

$$\begin{aligned} &D_0 = f_0 \\ &D_s = D_{s-1}(1+\delta) - f_s \qquad s = 1, 2, \ldots, n \end{aligned} \tag{2b}$$

where δ is the contractual rate.

The NPV (NFV) measures the incremental profit of alternative (i) over (ii), referred to the whole length of the business. It is equivalent to an excess profit, though calculated for the entire length n of the business. What about the economic profit in a generic period s? That is, what is the difference, for any period s, of (i) compared with (ii)? The relation between the NFV and the notion of excess profit are strict. The excess profit coincides with a periodic NFV and both measure the difference between the *factual* profit and the *counterfactual* profit the entrepreneur would obtain if capital were invested in some other business (e.g. in another economic sector).

In microeconomic terms the entrepreneur may invest her money in the firm (business P) or in some other business (account C), hence the excess profit. In financial terms the investor may invest her money in the project (business P) or in some other investment opportunity (account C), hence the periodic NPV (NFV).

'Excess profit' Definition (EPD). *Excess profit is the periodic differential profit of (i) over (ii).*

We now show that Stewart's and Peccati's models are equivalent.[2]

Stewart's Formal Translation (SFT). *The excess profit in period s is the difference of the profit and the interest the investor foregoes (Cost of capital) if she undertakes the investment.*

Peccati's Formal Translation (PFT). *The excess profit in period s is the Net Present (or Final) Value, calculated at the opportunity cost of capital, of a (fictitious) uniperiodic business whose cash flows are $-w_{s-1}$ and $w_s + a_s$ at time $s-1$ and s respectively, $s = 1, \ldots, n$.*

Denote with EVA_s (Economic Value Added) the excess profit in period s. Then, resting on SFT,[3]

$$\mathrm{EVA}_s = (\mathrm{NOPAT} - \text{Cost of Capital}) \tag{3}$$

where NOPAT is the net operating profit after taxes. Resting on a zero-debt assumption, we also have NOPAT=ROE*E where ROE is the return on equity and E is the equity.

[2] We use the expression "Stewart's model" as a (most popular) synecdoche to include the set of those models developed in the residual-income literature since the 1890s.

[3] Subscripts are omitted for convenience.

Peccati's Theorem. *Let G_s be the periodic share of the* NFV *to be ascribed to period s. Then,*

$$G_s = w_{s\text{-}1}(x-i)(1+i)^{n-s} \tag{4}$$

Proof: Resting on PFT, we focus on a generic period s: The investor invests the sum $w_{s\text{-}1}$ at the beginning of the period and receives w_s+a_s at the end of the period. The NFV of such a business is

$$-w_{s\text{-}1}(1+i)^{n-s-1} + (w_s + a_s)(1+i)^{n-s}$$

Using (2a) the latter boils down to (4) . Summing for s we obtain $\sum_{s=1}^{n} G_s$=NFV (analogously for a levered business, using (2b)). •

It is now easy to show that (3) and (4) are formally equivalent.

Theorem 1.1. *Stewart's compounded* EVA$_s$ *coincides with Peccati's share G_s.*

Proof: (3) can be rewritten as

$$\text{EVA}_s = \text{ROE}{*}\text{E} - \text{Cost of Capital}.$$

Just think that, in Peccati's terms, E=$w_{s\text{-}1}$, ROE=x, Cost of Capital=iw_{s-1} and the relation between (3) and (4) will be straightforward:

$$G_s = \text{EVA}_s(1+i)^{n-s}$$ •

Remark 1.1: Theorem 1.1 shows that SFT and PFT are equivalent.

Pressacco and Stucchi (*op.cit.*, henceforth P&S) modify Peccati's model in three senses:

(a) They allow account C to evolve according to the following recurrence equation:

$$C_s = C_{s\text{-}1}(1 + i(C_{s\text{-}1})) + a_s \tag{5}$$

where

$$i(C_{s\text{-}1}) = \begin{cases} i_P & \text{if } C_{s\text{-}1} > 0, \\ i_N & \text{if } C_{s\text{-}1} < 0, \end{cases}$$

with $i_P \neq i_N$,[4] and allow for non-Soper businesss. The outstanding capital w_s at the internal rate $x(w_{s\text{-}1})$ is given by

$$w_0 = a_0$$
$$w_s = w_{s\text{-}1}(1 + x(w_{s\text{-}1})) - a_s$$

[4] P stands for "positive", N for "negative".

where

$$x(w_{s-1}) = \begin{cases} x_P & \text{if } w_{s-1} > 0, \\ x_N & \text{if } w_{s-1} < 0. \end{cases}$$

(I shall henceforth assume $x_P \neq x_N$. Further, all interest rates we will be dealing with are assumed to be nonzero).[5] $x(w_{s-1})$ is then such that

$$w_n = -a_0(1+x(w))^n + \sum_{s=1}^{n} a_s(1+x(w))^{n-s} = 0 \tag{6}$$

where

$$(1+x(w))^{n-s} := \prod_{k=s+1}^{n} (1+x(w_{k-1})).$$

(b) they assume $C_0 = -a_0$, or, in other terms, $E_0=0$
(c) they assume $D_s=0$ for every s.

So doing P&S generalize Peccati's model in the sense of Teichroew *et al.* (1965a, 1965b), but at the same time they limit the scope of application of their model by ruling out the case $E_0 \neq 0$ and the case $D_s \neq 0$. In P&S's model the NFV for P is

$$\text{NFV} = -a_0(1+i(C))^n + \sum_{s=1}^{n} a_s(1+i(C))^{n-s} \tag{7}$$

where

$$(1+i(C))^{n-s} := \prod_{k=s+1}^{n} (1+i(C_{k-1})).$$

Note that in P&S model $i(C_{s-1})$ cannot be given the economic interpretation of an opportunity cost of capital as it represents a genuine rate (of cost or return depending on the sign of C_{s-1}). The main result of P&S (*op.cit.*, Theorem 6.2) is here summarized:

P&S Theorem. *Peccati's model can be generalized in*

$$\text{NFV} = \sum_{s:w_{s-1}>0}^{n} w_{s-1}(x_P - i_N)(1+i(C))^{n-s} + \sum_{s:w_{s-1}<0}^{n} w_{s-1}(x_N - i_P)(1+i(C))^{n-s} \tag{8a}$$

if and only if

$$x(w_{s-1}) = x_P \qquad \textit{iff} \qquad i(C_{s-1}) = i_N. \tag{8b}$$

[5] I shall never define the value of a two-valued rate when its argument is zero. In this case, we can pick whatever value we want.

It should now be clear that P&S may be formally seen as a generalized EVA model, thanks to equivalence of Peccati's and Stewart's model.

2. Systemic Value Added

In this section a decomposition model is offered differing in various aspects from the previous ones. While the previous models rest on financial arguments, the model here presented rests on economic reasoning and takes a microeconomic perspective. For the sake of notational convenience, I shall dwell on the case where D_s=0 for every s (see Magni, 2003, for some hints on the opposite case).

The argument goes as follows. Let us focus on time 0. The entrepreneur calculates the excess profit by computing the profit and subtracting all costs, among which the opportunity cost. We could say that the owner of the firm compares the profit she obtains from the firm and the profit she would obtain if the capital invested in the firm were invested in another alternative (in our case, account C). At time s the entrepreneur's wealth E_s is given by the sum of C_s and the outstanding capital w_s; with the alternative line of action the entrepreneur's wealth E^s will be given by E_0 plus the interest yielded by account C. The two alternative wealths are governed by two different dynamic systems:

$$\begin{aligned} C_0 &= E_0 - w_0 \\ w_0 &= a_0 \\ C_s &= C_{s-1}(1+i) + a_s \\ w_s &= w_{s-1}(1+x) - a_s \\ E_s &= C_s + w_s = E_{s-1} + xw_{s-1} + iC_{s-1} \qquad s \geq 1 \end{aligned} \tag{10a}$$

for case (i),

$$\begin{aligned} C^0 &= E_0 \\ C^s &= C^{s-1}(1+i) \\ E^s &= C^s = E^{s-1}(1+i) \qquad s \geq 1 \end{aligned} \tag{10b}$$

for case (ii). Given (EPD) and (10), an alternative interpretation of the notion of excess profit is here adopted and the following formal translation is offered:

Systemic Formal Translation (SYFT). *The excess profit in period s is the incremental income of (i) over (ii), based on the comparison of the investor's alternative wealths.*

I name such an excess profit *periodic Systemic Value Added* (SVA$_s$). We have then the following:

Systemic Theorem. *Excess profit is*

$$\mathrm{SVA}_s = xw_{s-1} - i\left(C^{s-1} - C_{s-1}\right) \tag{11}$$

Proof: Adopting SYFT we have to compute the profit from (10a) and the alternative profit from (10b), then take the difference between the two:

$$\mathrm{SVA}_s = \left[(E_s - E_{s-1}) - (E^s - E^{s-1})\right] = (C_s + w_s - C_{s-1} - w_{s-1}) - \left(C^s - C^{s-1}\right)$$

which represents the differential net profit of (i) over (ii) to be ascribed to period s, that is the excess profit generated by business P in period s. Using (10) such a residual income boils down to (11). •

The following result shows that, *in overall terms*, the SVA model is equivalent to the models of Stewart and Peccati (henceforth STEP):

Corollary 2.1 *Let* SVA:=$\sum_{s=1}^{n}$ SVA$_s$. *Then* SVA=NFV.

Proof: Summing for s all SVA$_s$ we have

$$\mathrm{SVA} = \sum_{s=1}^{n}\left((E_s - E_{s-1}) - (E^s - E^{s-1})\right) = E_n - E^n. \tag{12}$$

The conclusion follows by Definition 1.1. •

While coinciding in overall terms, STEP's model and the SVA model give rise to different partitions, for

$$G_s \neq xw_{s-1} - i\left(C^{s-1} - C_{s-1}\right)$$

$$\mathrm{EVA}_s \neq xw_{s-1} - i\left(C^{s-1} - C_{s-1}\right).$$

Remark 2.1: In the SVA model the entrepreneur's wealth is regarded as a dynamic system. Such a model is perhaps more economic than financial, in the following sense: Financial economists compare different rates of return applied to the same capital; for example, if the initial capital is 100 and the rate of return of business P is 10% per period, then the capital at time 1 is $110 = 100 + 10$, so the profit in the second period is 11=0.1(110). If 6% is the rate of return of the alternative opportunity, the excess profit is $11 - 0.06(110) = 6.6$. In a microeconomic outlook we should consider that if the alternative opportunity were followed, the capital invested in the second period would be different from 110. Given the alternative 6% rate of return, if the entrepreneur invested at time 0 a capital of 100 in the alternative opportunity, then the capital at time 1 would be $106 = 100 + 0.06(100)$, not 110. So the excess profit is, in this perspective, $4.64 = 11 - 0.06(106)$ instead of 6.6. In other terms, 106 is the capital foregone by the entrepreneur if she accepts to invest in P. It is a lost (unrecovered) capital.

Remark 2.2: We can see things in this way: In the SVA model the two alternative net worths at the beginning of period s can be written as

$$
\begin{aligned}
E_{s-1} &= C_{s-1} + w_{s-1} \\
E^{s-1} &= C_{s-1} + (C^{s-1} - C_{s-1}).
\end{aligned}
$$

Now, the term C_{s-1} is shared by both alternatives, so the differential terms are represented by the second addends. In the first case w_{s-1} yields profit at a rate x; in the second case $(C^{s-1} - C_{s-1})$ yields profit at a rate i. The amount $(C^{s-1} - C_{s-1})$ is the capital *lost* by the investor who chooses the project, and $i(C^{s-1} - C_{s-1})$ is the corresponding income lost by the investor: It is an opportunity cost, given that, in a systemic perspective, if the investor invested in account C (at time 0) rather than undertaking the project she would have, at time $s-1$, a $(C^{s-1} - C_{s-1})$ surplus in her account C. As one can see, all depends on the concept of opportunity cost, which, to Peccati and Stewart, coincides with iw_{s-1}, whereas, in a systemic perspective, it is represented by the lost income $i(C^{s-1} - C_{s-1})$. Owing to this interpretation, the systemic perspective may be said to induce a *lost-capital* paradigm of residual income (see Magni, 2009). The difference between the two models is thus not merely a mathematical one, but a cognitive (and economic) one.

A first interesting relation between SVA_s and EVA_s is now provided.

Proposition 2.1. STEP*'s model and the* SVA *model are such that*

$$
\text{SVA}_1 = \text{EVA}_1 \qquad \text{and} \qquad \text{SVA}_s = \text{EVA}_s + i\left(\sum_{k=1}^{s-1} \text{SVA}_k\right) \quad s > 1. \tag{13}
$$

Proof: The first relation in (13) is obvious, since $C^0 - C_0 = a_0 = w_0$. As for the second equality, using (2) we have

$$
\left(C^{s-1} - C_{s-1}\right) = w_{s-1} - \text{EVA}_1(1+i)^{s-2} - \text{EVA}_2(1+i)^{s-3} - \cdots - \text{EVA}_{s-2}(1+i) - \text{EVA}_{s-1}.
$$

Substituting in (11) we obtain

$$
\text{SVA}_s = \text{EVA}_s + \sum_{k=1}^{s-1} i\text{EVA}_k(1+i)^{s-1-k}. \tag{14}
$$

By induction,

$$
\sum_{k=1}^{s} \text{SVA}_k = \sum_{k=1}^{s} \text{EVA}_k(1+i)^{s-k} \tag{15a}
$$

for every $s \geq 1$.[6] Using now (15a) in (14) we get to (13). •

To clarify the above result let us decompose business P by means of G_s and SVA, where we assume, for the sake of convenience, n=3.

$$\begin{aligned} G_1 &= \text{EVA}_1(1+i)^2 & \qquad \text{SVA}_1 &= \text{EVA}_1 \\ G_2 &= \text{EVA}_2(1+i) & \text{SVA}_2 &= \text{EVA}_2 + i\text{SVA}_1 \\ G_3 &= \text{EVA}_3 & \text{SVA}_3 &= \text{EVA}_3 + i\text{SVA}_1 + i\text{SVA}_2 \end{aligned} \tag{16a}$$

or

$$\begin{aligned} G_1 &= \text{EVA}_1 + (i\text{EVA}_1) + (i\text{EVA}_1 + i^2\text{EVA}_1) & \qquad \text{SVA}_1 &= \text{EVA}_1 \\ G_2 &= \text{EVA}_2 + (i\text{EVA}_2) & \text{SVA}_2 &= \text{EVA}_2 + (i\text{EVA}_1) \\ G_3 &= \text{EVA}_3 & \text{SVA}_3 &= \text{EVA}_3 + (i\text{EVA}_1 + i^2\text{EVA}_1) \\ & & &\quad + (i\text{EVA}_2) \end{aligned} \tag{16b}$$

For STEP's model the idea is the following: EVA_1, EVA_2, EVA_3 are the three shares for period 1, 2, 3 respectively. As this is money referred to the dates 1, 2, 3, respectively, the basic principles of financial calculus force the evaluator to compound (or discount) flows to take time into consideration. After capitalization (and only after) the evaluator may sum the three shares. Conversely, in the light of our systemic perspective the decision maker can construct, in a gradual way, the three shares of the SVA. The first share is EVA_1, which exactly represents the difference between what the investor receives in the first period and what she would receive should she decide to forego the business opportunity and invest her funds at the opportunity cost of capital i. In the second period the difference between what she receives and what she would receive takes into account that, in addition to EVA_2, the first share yields differential interest equal to $i\text{EVA}_1$ ($=i\text{SVA}_1$). Iterating the argument, the third share considers the return gained on $i\text{EVA}_1$ as well as the return on the two first shares EVA_1 and EVA_2, which are produced just in the third period. Financially speaking, we can interpret every SVA_s as a capital invested at time s, yielding linear interest at the rate i until

[6] In particular, setting s=n, we have

$$\text{SVA} = \sum_{k=1}^{n} \text{SVA}_k = \sum_{k=1}^{n} \text{EVA}_k(1+i)^{s-k} = \text{NFV} \tag{15b}$$

n, for a total interest of $(i(n-s)\mathrm{SVA}_s)$ each. In fact, we can easily check that

$$\begin{aligned}
\mathrm{NFV} = \mathrm{SVA} &= \sum_{s=1}^{n} \mathrm{SVA}_s \\
&= \sum_{s=1}^{n} \mathrm{EVA}_s + \sum_{s=1}^{n} i \left(\sum_{h=1}^{s-1} \mathrm{SVA}_h \right) \\
&= \sum_{s=1}^{n} \mathrm{EVA}_s + \sum_{s=1}^{n-1} i(n-s)\mathrm{SVA}_s .
\end{aligned}$$

On the contrary, in STEP's model G_1 embodies the term $i\mathrm{EVA}_1$ which is instead generated in the second period, and comprehends $i\mathrm{EVA}_1 + i^2\mathrm{EVA}_1$ which in turn is related to the third period. Further, G_2 includes $i\mathrm{EVA}_2$, which relates to period 3, but lacks the term $i\mathrm{EVA}_1$ (previously embodied in G_1). Finally, the third share G_3 forgets the return on previous periods' shares.

Now we can extend the SVA model allowing for two-valued rates i and x depending on the sign of C_{s-1} and w_{s-1} respectively, as in P&S's model, while ruling out P&S restrictive assumption E_0=0.

Generalized Systemic Theorem. Let $i(C^{s-1})$ be defined so that $i(C^{s-1})=i_P$ if $C^{s-1}> 0$, $i(C^{s-1})=i_N$ if $C^{s-1}< 0$; let $i(C_{s-1})$ and $x(w_{s-1})$ be defined as in P&S's model. Then, the excess profit of business P is

$$\mathrm{SVA}_s = x(w_{s-1})w_{s-1} + i(C_{s-1})C_{s-1} - i(C^{s-1})C^{s-1}. \tag{17}$$

Proof: Straightforward by SYFT, since

$$\begin{aligned}
E_s &= E_{s-1} + i(C_{s-1})C_{s-1} + x(w_{s-1})w_{s-1} \\
E^s &= E^{s-1} + i(C^{s-1})C^{s-1}.
\end{aligned}$$

•

Remark 2.3: We have

$$\begin{aligned}
E_n &= (E_0 - a_0)(1 + i(C))^n + \sum_{s=1}^{n} a_s(1 + i(C))^{n-s} \\
E^n &= E_0(1 + i(C^0))^n = E_0(1 + i(E_0))^n
\end{aligned}$$

and

$$\begin{aligned}
\text{NFV} &= E_n - E^n \\
&= E_0\Big(\big(1+i(C)\big)^n - \big(1+i(E_0)\big)^n\Big) - a_0\big(1+i(C)\big)^n + \sum_{s=1}^{n} a_s\big(1+i(C)\big)^{n-s} \\
&= \sum_{s=1}^{n} \text{SVA}_s \\
&= \text{SVA}
\end{aligned} \tag{18}$$

Note that picking E_0=0 (and therefore C_0=$-a_0$) the NFV reduces to (7) as in P&S's model.

Remark 2.4: Note that the model here offered is not only more general than P&S's (account C is allowed to take on whatsoever value at time 0), it is actually alternative: Even if we assume C_0=$-a_0$ the periodic shares do not coincide.[7]

Remark 2.5 The decomposition we have arrived to is different from STEP's and P&S's model since it relies on a different interpretation of the notion of residual income: SYFT is based on incremental income computed as difference between alternative net profits, related to (i) and (ii) respectively. In other words, STEP's and P&S's models are based on the idea that at each time s the investor has the alternative of either investing w_{s-1} in the business or investing it at the rate i. Instead, the SVA model is based on the idea that we should start from wealth and periodic income: The decision maker has two alternatives at time 0; each of the two courses of action determines a different path for the financial dynamic system: Excess profit is then the difference between alternative profits drawn from the alternative dynamic systems.

This seems to be a striking result: We are capable of finding an alternative way of measuring excess profit, significant from an economic point of view. Therefore SYFT seems to constitute an enrichment in a firm's economic analysis, in that we have more than one way to formally translating the concept of excess profit. But there is much more than this. The systemic lost-capital residual income enjoys an aggregation property which is very important. Suppose that residual income is used for valuing an asset: By making use of the systemic lost-capital paradigm we have, from Corollary 2.1,

$$\frac{1}{(1+i)^n}\sum_{s=1}^{n}\text{SVA}_s = \text{NPV}$$

Because NPV=value of the asset $-$ cost, we have (remembering that cost=a_0)

$$\text{value of the asset} = a_0 + \frac{1}{(1+i)^n}\sum_{s=1}^{n}\text{SVA}_s.$$

[7] Note also that if i_P=i_N the NFV boils down to (1).

Contrast the above equation with the standard equation

$$\text{value of the asset} = a_0 + \sum_{s=1}^{n} \frac{\text{EVA}_s}{(1+i)^s}.$$

For valuation purposes in real life, every SVA_s and every EVA_s is a forecast. In the EVA paradigm, the asset's value is sensitive to forecasting errors in the computation of EVA_s; in the systemic paradigm, the asset's value is not sensitive to forecasting errors if the grand total residual incomes is correct. That is, suppose $\{\text{SVA}_s\}_1^n$ is the sequence of correct residual incomes and assume that $\{y_s\}_1^n$ is a permutation of the sequence (this corresponds to an error of imputation to periods of residual incomes): The resulting asset's value does not change because $\sum_{s=1}^n \text{SVA}_s = \sum_{s=1}^n y_s$. Even if the forecasted value for the SVA_s does not constitute a permutation of the correct sequence, the asset's value is not affected as long as the grand total remains unvaried. As a byproduct, this implies that forecasting with the systemic model is easier: One may rest on an average residual income and multiply by the number of periods to obtain the (NPV and) the asset's value, rather than forecasting each and every residual income EVA_s and discounting it with the discount factor $(1+i)^{-s}$. Putting it in other terms, the SVA_s's aggregate in a value sense, as opposed to the EVA_s's, which aggregate in a cash-flow sense (see also Magni, 2009).

3. The shadow business

I now introduce the concept of *shadow business*, assuming again i and x are constant.

Definition 3.1 *Let* $\Phi_k(y) := y(1+y)^{s-1-k}$ *and* $[\Phi_k]_a^b := \Phi_k(b) - \Phi_k(a)$ *for* $k = 0, 1, \ldots, s-1$. *Business* $\overline{P}$ *is said to be the shadow business of* P *if its cash flows are such that*

$$\overline{P} = (-\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_n)$$

where $\overline{a}_0 = a_0$ *and*

$$\overline{a}_s = a_s + a_0 \, [\Phi_0]_i^x + \sum_{k=1}^{s-1} a_k \, [\Phi_k]_x^i \qquad s = 1, \ldots, n.$$

Lemma 3.1 *Let* $\overline{w}_s := C^s - C_s$ *for* $s \geq 0$. *Then* $\overline{w}_0 = \overline{a}_0 = a_0$ *and*

$$\overline{w}_s = \overline{w}_{s-1}(1+i) - a_s = a_0(1+i)^s - \sum_{k=1}^{s} a_k (1+i)^{s-k} \qquad s \geq 1 \tag{19}$$

Proof: Use (10). •

Lemma 3.2 *Business P's cash-flow and its shadow $\overline{P}$'s cash-flow differ by the* SVA$_s$ *of business P, that is*

$$\mathrm{SVA}_s = \overline{a}_s - a_s \qquad \textit{for} \quad s \geq 1.$$

Proof: We have

$$\begin{aligned}
\overline{a}_s - a_s &= [\Phi_0]_i^x + \sum_{k=0}^{s-1} [\Phi_k]_x^i \\
&= a_0\Phi_0(x) - \sum_{k=0}^{s-1} a_k\Phi_k(x) - a_0\Phi_0(i) + \sum_{k=0}^{s-1} a_k\Phi_k(i) \\
&= a_0\, x(1+x)^{s-1} - \sum_{k=0}^{s-1} a_k\, x(1+x)^{s-k-1} - \left(a_0\, i(1+i)^{s-1} - \sum_{k=0}^{s-1} a_k\, i(1+i)^{s-1-k}\right) \\
&= xw_{s-1} - i(C^{s-1} - C_{s-1}) \\
&= [\text{by Systemic Theorem}] = \mathrm{SVA}_s
\end{aligned}$$

•

Proposition 3.1. *Let $\overline{x} := x\,\frac{w_{s-1}}{\overline{w}_{s-1}}$. Then*

$$\overline{w}_s = \overline{w}_{s-1}(1+\overline{x}) - \overline{a}_s \qquad \textit{for} \quad s \geq 1 \tag{20}$$

Proof:

$$\begin{aligned}
\overline{w}_s &= [\text{by Lemma 3.1})] = \overline{w}_{s-1} - a_s + i\overline{w}_{s-1} \\
&= [\text{by definition of } \overline{w}_s] = \overline{w}_{s-1} - a_s + i(C^{s-1} - C_{s-1}) \\
&= \overline{w}_{s-1} - a_s + i(C^{s-1} - C_{s-1}) + xw_{s-1} - xw_{s-1} \\
&= [\text{by Systemic Theorem}] = \overline{w}_{s-1} + xw_{s-1} - a_s - \mathrm{SVA}_s \\
&= [\text{by Lemma 3.2}] = \overline{w}_{s-1} + xw_{s-1} - \overline{a}_s \\
&= \overline{w}_{s-1}(1+\overline{x}) - \overline{a}_s.
\end{aligned}$$

•

Remark 3.1: Proposition 3.1 allows us to interpret $\overline{w}_s$ as the *business balance* of $\overline{P}$ at the rate $\overline{x}$, so that the concept of shadow business enables us to connect the SVA model to STEP's model:

Shadow Theorem. *Let* $\overline{\mathrm{EVA}}_s$ *be the Economic Value Added of $\overline{P}$. Then we have*

$$\overline{\mathrm{EVA}}_s = \mathrm{SVA}_s. \tag{21}$$

Proof: Let us apply STEP's arguments (and thus SFT and PFT) to the shadow of P. At the beginning of period s, the investor invests $\overline{w}_{s-1}$ and at the end of that period receives the sum $\overline{w}_s + \overline{a}_s$. So doing she renounces to the opportunity of investing that sum at the rate of interest i. She therefore foregoes the sum $\overline{w}_{s-1}(1+i)$. The Economic Value Added of $\overline{P}$ is

$$\begin{aligned}\overline{\mathrm{EVA}}_s &= -\overline{w}_{s-1}(1+i) + \overline{w}_s + \overline{a}_s \\ &= [\text{by (20)}] = -\overline{w}_{s-1}(1+i) + (\overline{w}_{s-1}(1+\overline{x}) - \overline{a}_s) + \overline{a}_s \\ &= \overline{w}_{s-1}(\overline{x}-i) \\ &= xw_{s-1} - i\overline{w}_{s-1} \\ &= xw_{s-1} - i\left(C^{s-1} - C_{s-1}\right) \\ &= \mathrm{SVA}_s.\end{aligned}$$

•

Remark 3.2: As one can note we have been able to retrieve STEP's model and adjust for a *systemic* partition of the Net Final Value of P. We discover an interesting result: If we are to partition the NFV of P under a systemic perspective, we can indeed use the concept of Economic Value Added as it is introduced by Stewart and Peccati, provided that we apply it to the shadow business $\overline{P}$ and do not capitalize the Economic Value Added so obtained.

Remark 3.3: In reframing the evaluation process we have applied STEP's argument to business $\overline{P}$. Therefore, moving from P to $\overline{P}$, we *shift* from PFT (and SFT) to SYFT. This result shows that the SVA model can be seen as an EVA model, provided that we compute the EVA of the shadow business and forget capitalization. Actually, the shadow business is a tool which enables us to shift from the interpretation of excess profit currently available in the literature to the new systemic interpretation here proposed.

The following section keeps on analyzing the relations among all models presented, but in a general setting and further results pertinent to the four models presented are shown, so as to embody all of them in a systemic view.

Henceforth we shall rest on a business P with internal pair (x_P, x_N) depending on the sign of the outstanding capital (x_P if positive, x_N if negative, as usual) and an account C with pair (i_P, i_N) depending on the sign of C (i_P if positive, i_N if negative, as usual). Letting $\overline{w}_s := C^s - C_s$, as before, we now have

$$\overline{w}_s = C^{s-1}(1 + i(C^{s-1})) - C_{s-1}(1 + i(C_{s-1})).$$

The shadow business $\overline{P}$ is now defined as in Definition 3.1, with $i(C_{s-1})(1+i(C))^{s-1-k}$ and $x(w_{s-1})(1+x(w))^{s-1-k}$ replacing $i(1+i)^{s-1-k}$ and $x(1+x)^{s-1-k}$ respectively. We will also make use of the rate $\overline{x}(\overline{w}_{s-1})$, defined as follows:

$$\overline{x}(\overline{w}_{s-1}) = \begin{cases} \overline{x}_P & \text{if } \overline{w}_{s-1} > 0 \\ \overline{x}_N & \text{if } \overline{w}_{s-1} < 0 \end{cases}$$

where $\overline{x}_N{:=}x_N\frac{w_{s-1}}{\overline{w}_{s-1}}$ and $\overline{x}_P{:=}x_P\frac{w_{s-1}}{\overline{w}_{s-1}}$.

4. The SVA Theorems

Definition 4.1: *A pair (i_P, i_N) is said to be a twin-pair if, for all s, $i(C^s)=i(C_s)$*

Definition 4.2: *A pair (i_P, i_N) is said to be an i_P-twin-pair if it is a twin-pair and $i(C_s)=i_P$. A pair (i_P, i_N) is said to be an i_N-twin-pair if it is a twin-pair and $i(C_s)=i_N$.*

Definition 4.3: *$\overline{P}$ is said to be a Soper business if, for all s, $\overline{x}(\overline{w}_{s-1}) = \overline{x}_P$. P is said to be a Soper business if for all s $x(w_{s-1})=x_P$*

Definition 4.4: *The shadow pair $(\overline{x}_P, \overline{x}_N)$ and the internal pair (x_P, x_N) are said to be parallel if, for all s,*

$$x(w_{s-1}) = x_P \qquad \textit{iff} \qquad \overline{x}(w_{s-1}) = \overline{x}_P.$$

Proposition 4.1. *If for all s C_s and C^s are both nonnegative or both nonpositive, then (i_P, i_N) is a twin-pair.*
Proof: From Definition 4.1 (and pointing out that $i(0)$ can be fixed *ad libitum*). •

Proposition 4.2. *If (i_P, i_N) is a twin-pair and there exists some s such that C_s and C^s do not have the same sign, then (i_P, i_N) is both i_P-twin and i_N-twin.*
Proof: The assumptions imply $i_P{=}i_N$. •

Remark 4.1: In Peccati's model (i_P, i_N) is both i_P-twin and i_N-twin.

Proposition 4.3. *If $E_0{=}0$, then (i_P, i_N) is a twin-pair and $C_s{=}{-}\overline{w}_s$ for all s.*
Proof: We have $C^s{=}0$ for all s and $\overline{w}_s{:=}C^s - C_s{=}{-}C_s$ for all s. Further, we have that $C^s{=}0$ for all s implies that, for all s, C^s and C_s are both nonnegative or both nonpositive, whence (i_P, i_N) is a twin-pair (Proposition 4.1). •

Proposition 4.4. *If (i_P, i_N) is an i_P-twin-pair, then $E_0{\neq}0$.*
Proof: If it were $E_0{=}0$, it would be $C_0{=}{-}a_0{<}0$, which contradicts the assumption. •

Proposition 4.5. *Suppose $E_0{=}0$. Then $\overline{P}$ is a Soper business if and only if (i_P, i_N) is an i_N-twin-pair.*
Proof: If $E_0{=}0$ then $C_s{=}{-}\overline{w}_s$ for all s and (i_P, i_N) is twin (Proposition 4.3). Then, if $\overline{P}$ is a Soper business, $C_s{\leq}0$ and therefore $i(C_s){=}i_N$ for all s. Conversely, if (i_P, i_N) is i_N-twin then $C_s{\leq}0$ for all s and therefore $\overline{w}_s{\geq}0$ for all s. Hence $\overline{x}(\overline{w}_{s-1}){=}\overline{x}_P$ for all s. •

Proposition 4.6. *If both P and $\overline{P}$ are Soper business, then the internal pair and the shadow pair are parallel. In particular, $x(w_{s-1}){=}x_P$ and $\overline{x}(\overline{w}_{s-1}){=}\overline{x}_P$.*
Proof: From Definitions 4.3 and 4.4. •

Proposition 4.7. *Suppose the shadow pair and the internal pair are parallel. Then P is a Soper business if and only if $\overline{P}$ is a Soper business.*

Proof: From Definitions 4.3 and 4.4. •

Theorem (SVA1). *If (i_P, i_N) is a twin-pair and the shadow pair and the internal pair are parallel, then*

$$\begin{aligned}\mathrm{SVA}_s &= \overline{w}_{s\text{-}1}\left(\overline{x}(\overline{w}_{s\text{-}1}) - i(C_{s\text{-}1})\right)\\ &= \overline{w}_{s\text{-}1}(\overline{x}_P - i_N)^{s_\sigma(1-s_\tau)}(\overline{x}_N - i_P)^{(1-s_\sigma)s_\tau}(\overline{x}_P - i_P)^{s_\sigma s_\tau}(\overline{x}_N - i_N)^{(1-s_\sigma)(1-s_\tau)}\end{aligned}$$

$$\text{for every } s, \tag{22a}$$

where s_τ=1 if $C_{s\text{-}1}$ is positive, s_τ=0 if $C_{s\text{-}1}$ is negative, s_σ=1 if $\overline{w}_{s\text{-}1}$ is positive, s_σ=0 if $\overline{w}_{s\text{-}1}$ is negative. Summing for s we have

$$\mathrm{SVA} = \mathrm{NFV} \tag{22b}$$

or, more explicitly,

$$\begin{aligned}\mathrm{SVA} = &\sum_{s:\overline{w}_{s\text{-}1}>0, C_{s\text{-}1}<0}^{n} \overline{w}_{s\text{-}1}(\overline{x}_P - i_N) + \sum_{s:\overline{w}_{s\text{-}1}<0, C_{s\text{-}1}>0}^{n} \overline{w}_{s\text{-}1}(\overline{x}_N - i_P)\\ &+ \sum_{s:\overline{w}_{s\text{-}1}>0, C_{s\text{-}1}>0}^{n} \overline{w}_{s\text{-}1}(\overline{x}_P - i_P) + \sum_{s:\overline{w}_{s\text{-}1}<0, C_{s\text{-}1}<0}^{n} \overline{w}_{s\text{-}1}(\overline{x}_N - i_N)\end{aligned}$$

Further

$$\mathrm{SVA}_s = \overline{\mathrm{EVA}}_s \qquad \text{for every } s. \tag{22c}$$

Proof: For the sake of convenience I shall label some propositions with conventional notations:

A_1: (i_P, i_N) is a twin-pair

A_2: the shadow pair $(\overline{x}_P, \overline{x}_N)$ and the internal pair (x_P, x_N) are parallel

A_3: for all s, $i(C_{s\text{-}1})C_{s\text{-}1} - i(C^{s\text{-}1})C^{s\text{-}1} = -i(C_{s\text{-}1})\overline{w}_{s\text{-}1}$

A_4: $x(w_{s\text{-}1})w_{s\text{-}1} = \overline{x}(\overline{w}_{s\text{-}1})\overline{w}_{s\text{-}1}$

A_5: SVA_s=$\overline{w}_{s\text{-}1}(\overline{x}(\overline{w}_{s\text{-}1}) - i(C_{s\text{-}1}))$

A_6: $\overline{\mathrm{EVA}}_s = -\overline{w}_{s\text{-}1}(1 + i(C_{s\text{-}1})) + \overline{w}_s + \overline{a}_s = -\overline{w}_{s\text{-}1}(1 + i(C_{s\text{-}1})) + \overline{w}_{s\text{-}1}(1 + \overline{x}(\overline{w}_{s\text{-}1}))$

A_1 implies A_3, A_2 implies A_4. A_3, A_4 and (17) imply A_5, which in turn implies (22a). Further, (22b) holds, due to (18).

Let us now calculate the shadow business's Economic Value Added ($\overline{\mathrm{EVA}}_s$). It is easy to see that

$$\overline{w}_s = \overline{w}_{s\text{-}1}(1 + \overline{x}(\overline{w}_{s\text{-}1})) - \overline{a}_s$$

since the shadow pair and the internal pair are parallel. We can then interpret $\overline{w}_s$ as $\overline{P}$'s business balance at time s at the rate $\overline{x}(\overline{w}_{s-1})$. Applying STEP's arguments we get to A_6. The latter coincides with A_5, so (22c) holds. •

Note that (22) tells us that for all s, one of the following holds:

$$\begin{array}{ll}
\overline{w}_{s-1}(\overline{x}_P - i_N) = \overline{\mathrm{EVA}}_s & \text{if } \overline{w}_{s-1}>0 \text{ and } C_{s-1}<0 \\
\overline{w}_{s-1}(\overline{x}_N - i_P) = \overline{\mathrm{EVA}}_s & \text{if } \overline{w}_{s-1}<0 \text{ and } C_{s-1}>0 \\
\overline{w}_{s-1}(\overline{x}_P - i_P) = \overline{\mathrm{EVA}}_s & \text{if } \overline{w}_{s-1}>0 \text{ and } C_{s-1}>0 \\
\overline{w}_{s-1}(\overline{x}_N - i_N) = \overline{\mathrm{EVA}}_s & \text{if } \overline{w}_{s-1}<0 \text{ and } C_{s-1}<0
\end{array}$$

Theorem (SVA2). *If $E_0=0$ and the shadow pair and the internal pair are parallel, then (22) holds, with $s_\tau = 1$ iff $s_\sigma = 0$.*
Proof: As before let us make use of the following conventions:

B_1: $E_0=0$

B_2: the shadow pair $(\overline{x}_P, \overline{x}_N)$ and the internal pair (x_P, x_N) are parallel

B_3: (i_P, i_N) is a twin-pair

B_4: for all s, $C_s=-\overline{w}_s$

B_5: $\overline{x}(\overline{w}_{s-1})=\overline{x}_P$ if and only if $i(C_{s-1})=i_N$

B_6: $\overline{x}(\overline{w}_{s-1})-i(C_{s-1})=(\overline{x}_P - i_N)$ or $\overline{x}(\overline{w}_{s-1})-i(C_{s-1})=(\overline{x}_N - i_P)$

B_7: $s_\tau=1$ if and only if $s_\sigma=0$

B_1 implies B_3 and B_4 (Proposition 4.3). B_2 and B_3 imply (22) (SVA1). B_4 implies B_5. B_5 implies B_6. B_6 and (22a) imply B_7. •

Proposition 4.8. *(22a) holds if and only if*

$$x(w_{s-1})w_{s-1} - \overline{x}(\overline{w}_{s-1})\overline{w}_{s-1} = C^{s-1}\left[i(C^{s-1}) - i(C_{s-1})\right] \qquad \textit{for every } s \tag{23}$$

Proof: (22a) holds if and only if

$$\overline{x}(\overline{w}_{s-1})\overline{w}_{s-1} - i(C_{s-1})\overline{w}_{s-1} = x(w_{s-1})w_{s-1} + i(C_{s-1})C_{s-1} - i(C^{s-1})C^{s-1}$$

whence

$$\begin{aligned}
x(w_{s-1})w_{s-1} - \overline{x}(\overline{w}_{s-1})\overline{w}_{s-1} &= i(C^{s-1})C^{s-1} - i(C_{s-1})\left[\overline{w}_{s-1} + C_{s-1}\right] \\
&= C^{s-1}\left[i(C^{s-1}) - i(C_{s-1})\right].
\end{aligned}$$

•

I now prove that the assumptions of SVA1 and SVA 2 are not necessary for (22) to hold, by providing a counterexample. Choose E_0=-30, n=2, a_0=700,a_1=850, x_N=0.35, x_P=0.3, i_N=0.15, i_P=0.0630434782608. We have then

$$\begin{aligned}
C^0 &= -30 < 0 & C^1 &= -30(1.15) = -34.5 < 0\\
C_0 &= -730 < 0 & C_1 &= -730(1.15) + 850 = 10.5 > 0\\
w_0 &= a_0 = 700 > 0 & w_1 &= 700(1.3) - 850 = 60 > 0\\
\overline{w}_0 &= C^0 - C_0 = 700 > 0 & \overline{w}_1 &= C^1 - C_1 = -45 < 0\\
i(C^0) &= i_N & i(C^1) &= i_N\\
i(C_0) &= i_N & i(C_1) &= i_P\\
x(w_0) &= x_P & x(w_1) &= x_P\\
\overline{x}(\overline{w}_0) &= \overline{x}_P = \frac{x_P w_0}{\overline{w}_0} & \overline{x}(\overline{w}_1) &= \overline{x}_N = \frac{x_N w_1}{\overline{w}_1}
\end{aligned}$$

and a_2 is univocally determined (=78). (23) holds, since

$$0.3 * 700 - 0.3 * 700 = -30\,[0.15 - 0.15]$$

for period 1, and

$$0.3 * 60 - (-45) * \frac{0.35 * 60}{-45} = -34.5\,[0.15 - 0.0630434782608]$$

for period 2. Further,

$$\mathrm{SVA}_1 = \overline{w}_0(\overline{x}_P - i_N)$$

$$\mathrm{SVA}_2 = \overline{w}_1(\overline{x}_P - i_N).$$

Therefore the conclusions of both SVA1 and SVA2 hold, i.e. (22a) holds, with s_τ=1 if and only if s_σ=0.

I have provided a counterexample which proves that the assumptions of both SVA1 and SVA2 are not necessary, since their conclusions hold, whereas neither of their assumptions holds: We have, in fact,

(#1) $E_0 \neq 0$
(#2) (i_P, i_N) is non-twin
(#3) the shadow pair and the internal pair are not parallel.

Remark 4.2: Note that (#2) implies (#1) (Proposition 4.3, *modus tollens*).

Proposition 4.9. *If (22) holds then (i_P, i_N) is twin if and only if the shadow pair and the internal pair are parallel.*

Proof: (23) holds (Proposition 4.8). Assume (i_P, i_N) is twin. Then the right-hand side of (23) must be zero for all s, which implies the same for the left-hand side, that is the shadow pair and the internal pair are parallel. Assume now the latter. Then the left-hand side of (23) is zero for all s, which implies the same for the right-hand side. Therefore, E_0 is zero or (i_P, i_N) is twin. Should the former of these two hold, then the latter is implied (Proposition 4.3). •

Proposition 4.9 enables us to prove that if (22) holds we cannot have (#2) without (#3) and vice versa. Thus, if we want to prove that the assumptions of SVA1 are not necessary we cannot invalidate only one of them.

Proposition 4.10. *Suppose that (22) holds alongside either (#2) or (#3). Then the other one also holds.*
Proof: Proposition 4.9 and Proposition 4.10 are tautologically equivalent. •

I restate here P&S Theorem, making explicit the implicit assumption E_0=0 and making use of the notion of *parallel* pair here introduced:

P&S Theorem. *Assume E_0=0. Then the* NFV *of P can be written as*

$$\text{NFV} = \sum_{s:w_{s-1}>0}^{n} w_{s-1}(x_P - i_N)(1+i(C))^{n-s} + \sum_{s:w_{s-1}<0}^{n} w_{s-1}(x_N - i_P)(1+i(C))^{n-s}$$

if and only if the shadow pair and the internal pair are parallel.

We are now ready to state the systemic counterpart of P&S Theorem.

Theorem (SVA3). *Assume E_0=0. Then (22) holds, with $s_\tau = 1$ iff $s_\sigma = 0$, if and only if the shadow pair and the internal pair are parallel.*

In particular, the NFV *of P can be written as*

$$\text{NFV} = \sum_{s:\overline{w}_{s-1}>0}^{n} \overline{w}_{s-1}(\overline{x}_P - i_N) + \sum_{s:\overline{w}_{s-1}<0}^{n} \overline{w}_{s-1}(\overline{x}_N - i_P).$$

Proof: Assume that, in addition to E_0=0, the shadow pair and the internal pair are parallel: Then (22) holds, with s_τ=1 if and only if s_σ=0 (SVA2). Conversely, assume that, in addition to E_0=0, (22) holds with s_τ=1 if and only if s_σ=0. Then (i_P, i_N) is a twin-pair (Proposition 4.3) and the shadow pair and the internal pair are parallel (Proposition 4.9).•

Theorem (SVA4). *If both P and $\overline{P}$ are Soper businesss and E_0=0, then (22) holds with s_σ=1 and s_τ=0 for all s.*
Proof: Let

D_1: E_0=0

D_2: P is a Soper business

D_3: $\overline{P}$ is a Soper business

D_4: the shadow pair $(\overline{x}_P, \overline{x}_N)$ and the internal pair (x_P, x_N) are parallel

D_5: (i_P, i_N) is an i_N-twin-pair

D_6: s_σ=1 and s_τ=0 for all s

D_2 and D_3 imply D_4 (Proposition 4.6). D_1 and D_3 imply D_5 (Proposition 4.5). D_1 and D_4 imply that, for all s, one of the following holds:

$$\mathrm{SVA}_s = \overline{w}_{s\text{–}1}(\overline{x}_P - i_N) \tag{24a}$$

$$\mathrm{SVA}_s = \overline{w}_{s\text{–}1}(\overline{x}_N - i_P) \tag{24b}$$

(SVA3). As D_5 holds, (24b) must be ruled out, and (24a) coincides with D_6. •

I restate here Proposition 6.1 of P&S (*op.cit.*, p.179) in our systemic parlance:

Proposition 4.11.1. *If E_0=0, (i_P, i_N) is an i_N-twin-pair, P is a Soper business, then*

$$\mathrm{NFV} = \sum_{s=1}^{n} w_{s\text{–}1}(x_P - i_N)(1 + i_N)^{n-s}.$$

I now prove the systemic counterpart of Proposition 4.11.1:

Proposition 4.11.2. *If E_0=0, (i_P, i_N) is an i_N-twin-pair, P is a Soper business, then the conclusion of* SVA4 *holds.*

In particular, we have

$$\mathrm{NFV} = \mathrm{SVA} = \sum_{s=1}^{n} \overline{w}_{s\text{–}1}(\overline{x}_P - i_N).$$

Proof: The first two hypotheses imply that $\overline{P}$ is a Soper business (Proposition 4.5). The latter, the first hypothesis and the third hypothesis are just SVA4's assumptions, so that (24a) holds. •

Remark 4.3: The two Propositions get back to a particular case of Peccati's model, in which E_0 is zero, P is assumed to be a Soper business and the value of account C is always negative. Even though, strictly speaking, they are not inconsistent each other in overall terms, it is clear that the periodic NFV's shares differ and that different perspectives are at work.

Remark 4.4: A striking result is that Proposition 6.1 of P&S (corresponding to our Proposition 4.11.1) can be easily proved if we make use of our systemic approach. The proof is straightforward, due to Proposition 4.11.2, (15b) and the following equalities:[8]

$$\overline{w}_{s\text{–}1}(\overline{x}_P - i_N) = \overline{\mathrm{EVA}}_s = \mathrm{SVA}_s$$
$$w_{s\text{–}1}(x_P - i_N) = \mathrm{EVA}_s.$$

[8] Actually, (15b) still holds with $i(C)$ replacing i.

Note also that the first two hypotheses in Propositions 4.11.1 and 4.11.2 imply that $\overline{P}$ is a Soper business. This suggests us that we can relax the first hypothesis:

Proposition 4.11.2a. *If (i_P, i_N) is an i_N-twin-pair and both P and $\overline{P}$ are Soper businesss, then the conclusion of* SVA4 *holds.*

In particular, we have

$$\text{NFV} = \text{SVA} = \sum_{s=1}^{n} \overline{w}_{s\text{–}1}(\overline{x}_P - i_N).$$

Proof: The first hypothesis implies (i_P, i_N) is a twin-pair, with $i(C_{s\text{–}1}){=}i(C^{s\text{–}1}){=}i_N$. The second and the third hypotheses imply that the shadow pair and the internal pair are parallel, with $x(w_{s-1}) = x_P$ and $\overline{x}(\overline{w}_{s-1}) = \overline{x}_P$ (Proposition 4.6). Hence, (22) holds with $s_\sigma{=}1$ and $s_\tau{=}0$ for all s. •

As for P&S's model, we have the following

Proposition 4.11.1a. *If (i_P, i_N) is an i_N-twin-pair and both P and $\overline{P}$ are Soper businesss, then*

$$\text{NFV} = \sum_{s=1}^{n} w_{s\text{–}1}(x_P - i_N)(1+i_N)^{n-s}.$$

Proof: We just have to make use of the systemic approach. The proof mirrors the argument in *Remark 4.4*, relying on Proposition 4.11.2a, (15b) and the equalities shown. •

Remark 4.5: We could further generalize Proposition 4.11.1a by removing the third assumption on $\overline{P}$ being a Soper business. The latter is essential only if we want to prove the Proposition via Proposition 4.11.2a. The first two hypotheses are actually sufficient to get to the conclusion, because (15b) holds regardless of being $\overline{P}$ a Soper business or not.[9]

Remark 4.6: On the basis of the latter Proposition's proof and *Remark 4.4* one may wonder whether we can use the systemic approach to prove all the results P&S have reached. The answer is yes but I will not dwell on it, leaving a thorough investigation for a next paper. I just give some hints for the proof of P&S Theorem. The proof is easy: We just have to use SVA3, (15b) and remember that

$$\begin{aligned}
\overline{\text{EVA}}_s = \text{SVA}_s = \overline{w}_{s\text{–}1}(\overline{x}_P - i_N) \quad \text{whenever} \quad \text{EVA}_s = w_{s\text{–}1}(x_P - i_N)\\
\overline{\text{EVA}}_s = \text{SVA}_s = \overline{w}_{s\text{–}1}(\overline{x}_N - i_P) \quad \text{whenever} \quad \text{EVA}_s = w_{s\text{–}1}(x_N - i_P).
\end{aligned}$$

[9] However, if P is a Soper business but $\overline{P}$ is not, the shadow pair and the internal pair are not parallel, as Proposition 4.7 indirectly suggests. This means we are not sure that

$$\overline{w}_{s\text{–}1}(\overline{x}_P - i_N) = \text{SVA}_s$$

for every s, so that Proposition 4.11.2a needs the "Soper condition" for both P and $\overline{P}$ to ensure its conclusion.

Conclusive Remarks

This paper shows that modelling excess profit (residual income) is not an unambiguous task and that more than one interpretation of this notion is possible. The existing models in the literature rely on the accounting and financial literature, which may be formally condensed in Stewart's (1991) model. The latter is shown to be equivalent to the NPV decomposition model developed by Peccati (1989). This paper proposes a different interpretation, according to which residual income is obtained by focusing on the wealth's diachronic evolution, that is, wealth is regarded as an economic dynamic system. The systemic outlook regards excess profit as the incremental income of one alternative over the other, computed on the basis of the alternative dynamic systems relative to the two courses of action. The two alternatives are (i) to invest in the business (ii) to invest in account C, or (i) to invest in the firm (ii) to invest in some other business; the excess profit is the difference between the firm's profit and the profit that could be earned by investing in some other business.

The two different interpretations give rise to different values of excess profit and this paper tries to shed light on the relations between the two and their differences as well. Differences in the notion of excess profit are differences in the notion of opportunity cost. Opportunity cost is the foregone return: In the current models such a foregone return is just the product of the rate i and the outstanding capital w_{s-1}, whereas in the SVA model the opportunity cost is given by the product of the rate i and the capital lost by the investor $\overline{w}_{s-1}=C^{s-1}-C_{s-1}$, which warrants the label "lost-capital" paradigm given by Magni (2009). We could say that STEP and P&S give us 'business-oriented' models, as opposed to the SVA, which is 'wealth-oriented'. This means that the opportunity cost in the former depends on the *business*'s evolution, whereas the latter depends on the *wealth*'s evolution. The evaluator willing to choose STEP's model is prone to accept the following argument: In period s, one can invest w_{s-1} at the rate x or at the rate i, so the difference $xw_{s-1}-iw_{s-1}$ is the excess profit. Conversely, the evaluator whose cognitive outlook adopts the SVA interpretation, is inclined to argue as follows: If alternative (i) is selected at time 0, then in period s one is renouncing to invest $\overline{w}_{s-1}$ at the rate i, since account C's value is C_{s-1} while it would be C^{s-1} should she select alternative (ii). So, the foregone return is the product of rate i and the lost capital $C^{s-1}-C_{s-1}$.

In financial economics, to speak of excess profit means to speak of the difference between the *factual* profit of the firm and the *counterfactual* profit the firm would earn if *that very factual capital* were invested in an alternative business. The systemic approach modifies this outlook and takes what seems a more genuinely economic outlook: The excess profit of the firm is the difference between the *factual* profit of the firm and the *counterfactual* profit the firm would earn if the same initial capital were invested in an alternative business, *taking into account that investing in the alternative business implies that the value of the capital invested in each period is different from what it would be if it were invested in alternative (i).* So, the counterfactual profit is different in the following periods not only because the rate of return is different, but also because the capital invested is different.

All this seems to be an interesting result: From a theoretical point of view, we find that the notion of excess profit is not unambiguous, since more than one translation is possible. We have two different interpretations of the same concept, therefore two different ways of measuring it. The SVA may be used in a capital-budgeting context, as well as a measure of a business's periodic performance and, in general, as a tool of corporate governance (e.g. for rewarding managers). Both aspects deserve attention and future researches may be addressed to studying in which situations one or the other perspective suits better the evaluator's needs, and to analysing more thoroughly the theoretical concept of excess profit, maybe finding that such a notion is a conventional one, with no way of determining an 'objective' excess profit. For valuation purposes, it is shown that the SVA model enjoys an aggregation property that enables one to neutralize possible forecasting errors due to an incorrect imputations of residual incomes. Such a property also favors computation of a firm's value, in that one can rest on the accounting notion of average residual income, based on the past history of the firm.

A third aspect seems full of implications, too, from a formal point of view. The two perspectives, though alternative, seem to be strictly connected: The concept of shadow business here introduced seems to provide the link between the two, so that a sort of 'duality' between business and evaluation techniques could be established: The result that the SVA$_s$ of business P coincides with the EVA$_s$ of its shadow business $\overline{P}$ may be iterated. In fact, P is the shadow business of some other business P'. So, the EVA of P is just the SVA of P'. But P' is in turn the shadow business of some other business and so on.

The model presented has been here generalized in the sense of Teichroew *et al.* and some results shed light on the concepts of twin-pair, parallel pairs, Soper business, and on the necessary and sufficient conditions for the two perspectives to be integrated. Future researches may be devoted to removing the zero-debt assumption and introducing multiple accounts playing the same role as account C (see Magni, 2003, for some hints).

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